Lesson 16: Observability

- Observability (DT)
- Observability (CT)
- Kalman standard form
Observability

- A state $x = q$ is said unobservable over $[0 \ T]$, if for every input $u$, the output $y_q$ obtained with initial condition $x(0) = q$ is the same as the output $y_0$ obtained with initial condition $x(0) = 0$.

- A dynamic system is said unobservable if it contains at least an unobservable state, observable otherwise.

- Note that observability can be established assuming zero input.

- The initial state $x(0)$ can be uniquely determined from input/output measurements iff the system is observable.
Discrete-time Observability

- Given a system described by the \( n \) state-space model

\[
\begin{align*}
x(k + 1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k) + Du(k)
\end{align*}
\]

- Suppose we are given \( u(k) \) and \( y(k) \) for \( 0 \geq t < T \).

\[
\begin{bmatrix}
y(0) \\
y(1) \\
\vdots \\
y(T - 1)
\end{bmatrix}
= \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{T-1}
\end{bmatrix}
\begin{bmatrix}
x(0) + \\
u_0 \\
u_1 \\
u(T - 1)
\end{bmatrix}
\]

\[
\begin{bmatrix}
D & 0 & 0 & \cdots & 0 \\
CB & D & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
CA^{T-2}B & CA^{T-3}B & \cdots & 0 \\
\end{bmatrix}
\]
Discrete-time Observability

Now the second term on the right (the forced response) is known, so we can subtract it from the vector of measured outputs to get

\[
\bar{y} = \begin{bmatrix}
\bar{y}(0) \\
\bar{y}(1) \\
\vdots \\
\bar{y}(T-1)
\end{bmatrix} = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{T-1}
\end{bmatrix} x(0) = \mathcal{O}_T x(0)
\]

\(\mathcal{O}_T\) is the \(T\)-step observability matrix
Observability (DT): Properties

- **Theorem**
  The set of states that is unobservable over $T$ steps is precisely $\mathcal{N}(\mathcal{O}_T)$, and is therefore a subspace.

- **Notice also that**
  \[
  \mathcal{N}(\mathcal{O}_k) \supseteq \mathcal{N}(\mathcal{O}_{k+1})
  \]
  \[
  \mathcal{N}(\mathcal{O}_n) = \mathcal{N}(\mathcal{O}_{n+l}), \quad l \geq n
  \]

- **Theorem**
  If $x(0) = \xi$ is unobservable over $n$ steps, then it is unobservable over any number of steps. Equivalently, the system is observable if and only if $\text{rank}(\mathcal{O}_n) = n$. 
We begin by defining the $k$-step observability Gramian as

$$Q_k = O_k^T O_K = \sum_{i=0}^{k-1} (A^i)^T C^T CA^i$$

The unobservable space over $k$ steps is evidently the nullspace of $Q_k$.

The system is observable if and only if $\text{rank}(Q_n) = n$
Continuous-time observability

- For continuous-time systems, the following are equivalent: $q$ is unobservable in time $T$ is unobservable in any time.
- Observability Gramian:

$$Q_t = \int_0^t (e^{A\tau})^T C^T C e^{A\tau} d\tau$$

- The system is then observable if and only if $\text{rank}(Q_t) = n$
Other results

- Essentially, observability results are similar to their reachability counterparts, when considering $(A^T, C^T)$ as opposed to $(A, B)$. (Duality Theorem)
- $(A, C)$ is unobservable if $Cv_i = 0$ for some (right) eigenvector $v_i$ of $A$.
- $(A, C)$ is unobservable if $(zI - A)C$ drops ranks for some $z = \lambda$, this $\lambda$ is an unobservabe eigenvalue for the system.
- The dual of controllability to the origin is constructability: same considerations as in the reachability/controllability case hold.
Standard form (Observability)

- We can construct a matrix $T_{\bar{o}}^{n\times o}$, whose columns span the nullspace of the observability matrix $O_n$.
- We may then construct $T_{\bar{o}}$ by selecting $(n - o)$ linearly independent vectors, such that $\text{rank}(T^{n\times n}) = \text{rank}[T_{\bar{o}} \ T_{\bar{o}}'] = n$.
- Since $T$ is invertible, we can perform a similarity transform to generate an equivalent system:
  \[
  AT = A[T_{\bar{o}} \ T_{\bar{o}}'] = T\tilde{A} = [T_{\bar{o}} \ T_{\bar{o}}'] \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix},
  \]
  \[
  CT = C[T_{\bar{o}} \ T_{\bar{o}}'] = \tilde{C} = [0 \ C_1]
  \]
- The presence of the zero blocks in the transformed system and output matrices follows from an argument similar to that used for the reachable canonical form coupled with the fact the the unobservable space is also $A$-invariant.
- $\text{rank}(\bar{O}_n) = \text{rank}(O_n)$
Lesson 17: Model-based Controller

- Feedback Stabilization
- Observers
- Ackerman Formula
- Model-based Controller
Feedback Stabilization

The state of a reachable system can be steered to any desired state in finite time, even if the system is unstable.

However, an open-loop control strategy depends critically on a number of assumptions:
1. Perfect knowledge of the model;
2. Perfect knowledge of the initial condition;
3. No input constraints.

It is necessary to use some information on the actual system state in the computation of the control input: i.e., feedback.

Feedback can also improve the performance of stable systems... but done incorrectly, can also make things worse, most notably, make stable systems unstable.
State Feedback

- Assume we can measure all components of a system’s state, i.e., consider a state-space model of the form \((A, B, I, 0)\) (the state is accessible).
- Assume a linear control law of the form \(u = Fx + v\).
- The closed-loop system model is \((A + BF, B, I, 0)\).
- Hence, it is clear that the closed-loop system is stable if and only if the eigenvalues of \(A + BF\) are all in the open left-half plane (CT) (or all inside the unit circle, (DT)).
Eigenvalue Placement

Theorem
There exists a matrix $F$ such that $det(\lambda I - (A + BF)) = \prod_{i=1}^{n}(\lambda - \mu_i)$ for any arbitrary self-conjugate set of complex numbers $\mu_1, \ldots, \mu_n \in \mathbb{C}$ if and only if $(A, B)$ is reachable.

Proof (necessity):
Suppose $\lambda_i$ is an unreachable mode, and let $w_i$ be the associated left eigenvector. Hence, $w_i^T A = \lambda_i w_i$, and $w_i B = 0$. Then,

$$w_i^T (A + BF) = w_i^T A + w_i^T BF = \lambda_i w_i^T + 0$$

i.e., $\lambda_i$ is an eigenvalue of $A + BF$ for any $F$!
Proof (sufficiency):
- Assuming the system is reachable, find a feedback such that the closed-loop poles are at the desired locations. We will prove this only for the single-input case.
- If the system is reachable, then w.l.o.g. we can assume its realization is in the controller canonical form: the coefficients of the characteristic polynomial are \( a_1, a_2, \ldots, a_n \).
- The coefficients of the closed-loop characteristic polynomial are \( (a_1 - f_1) \), etc..
- Just choose \( f_i = a_i - a_i^d \), \( i = 1, \ldots, n \).
Ackerman Formula

\[ F = -[0, 0, ..., 1] R_n^{-1} \alpha^d(A). \]
Observer

\[ y = Cx \]

\[
\begin{array}{c}
\dot{x} \\
\text{estimate of } x
\end{array}
\]

controller
Observer (Luenberger)

- What if we cannot measure the state?
- Design a **model-based observer**, i.e., a system that contains a simulation of the system, and tries to match its state.

\[
\hat{x}(k+1) = A\hat{x}(k) + Bu(k) - L(y(k) - \hat{y}(k)).
\]

- **Error dynamics**: \( e = x - \hat{x} = \tilde{x} : \)

\[
e(k+1) = Ax(k) + Bu(k) - A\hat{x}(k) - Bu(k) + L(y(k) - \hat{y}(k)) \\
= (A + LC)e(k)
\]

- Same results (dual) as for reachability.
Result

- **Theorem:**
  There exists a matrix $L$ such that
  $$\det(\lambda I - [A + LC]) = \prod_{i=1}^{n}(\lambda - \mu_i)$$
  for any arbitrary self-conjugate set of complex numbers $\mu_i$, $i = 1, \ldots, n$ if and only if $(C, A)$ is observable.

- **Ackerman formula:**
  $$L = \alpha^d(A)\theta_n^{-1}en$$
Model Based Controller
Model-based controller

We have

\[ x(k + 1) = \]
\[ Ax(k) + Bu(k) = \]
\[ Ax(k) + B(r(k) + F\hat{x}(k)) = \]
\[ Ax(k) + BF\hat{x}(k) + Br(k) = \]
\[ (A + BF)x(k) - BF\hat{x}(k) + Br(k) \]

Now, if the estimation error is \( \tilde{x} = x - \hat{x} \), then

\[ \tilde{x}(k + 1) \]
\[ = Ax(k) + Bu(k) - A\hat{x}(k) - Bu(k) + LC(x(k) - \hat{x}(k)) \]
\[ = (A + LC)\tilde{x}(k) \]

Then in summary the overall system is

\[
\begin{bmatrix}
  x \\
  \tilde{x}
\end{bmatrix}
(k + 1) =
\begin{bmatrix}
  A + BF & -BF \\
  0 & A + LC
\end{bmatrix}
\begin{bmatrix}
  x \\
  \tilde{x}
\end{bmatrix}
(k) +
\begin{bmatrix}
  B \\
  0
\end{bmatrix}

r(k)
Synthesis

- Poles of the closed-loop = c.l. poles of the controller ∪ c.l. poles of the observer.
- Separation principle: can design controller and observer independently!
- What if the system is not completely reachable or completely observable??
Detectability and Stabilizability

- A system is said **Stabilizable** if all the **unreachable** eigenvalues are stable \( (\lambda_{\bar{r}} \in C_s) \)
- A system is said **Detectable** if all the **unobservable** eigenvalues are stable \( (\lambda_{\bar{o}} \in C_s) \)
- If a system is Stabilizable and Detectable it is possible to design a model based controller, but the unreachable eigenvalues need to be in the set of desired eigenvalues.
Lesson 18: Minimal State-Space Realization

- Kalman Decomposition
- Interconnections and minimality
The Kalman Decomposition

Suppose we construct a transformation matrix
\[ T = [T_{r\bar{o}} \quad T_{ro} \quad T_{\bar{r}\bar{o}} \quad T_{\bar{r}o}] \] such that
1. The columns of \( T_{r\bar{o}} \) form a basis for \( \mathcal{R} \cap \bar{O} \), the subspace that is both reachable and unobservable
2. \( T_{ro} \) complements \( T_{r\bar{o}} \) in the reachable subspace, so that \( Ra[T_{r\bar{o}} \quad T_{ro}] = \mathcal{R} \)
3. \( T_{\bar{r}\bar{o}} \) complements \( T_{r\bar{o}} \) in the unobservable subspace, so that \( Ra[T_{r\bar{o}} \quad T_{\bar{r}\bar{o}}] = O \)
4. \( T_{\bar{r}o} \) complements the rests of the column to span \( \mathbb{R}^n \), so that \( T \) is invertible.

Perform a similarity transformation using \( T \)

The system \((\hat{A}, \hat{B}, \hat{C}, D)\) is said to be in Kalman Decomposed form
Kalman Decomposition
Minimality

- **Theorem**
  A state-space realization of a SISO transfer function $H$ is minimal iff it is reachable and observable.

- **Theorem**
  All minimal realizations of a given transfer function are similar to each other.

- **Note** the relation with Interconnected systems and pole/zero cancellation.

- **Note** the relation among internal and external stability.
Ex 7: Reachability and Observability, Model Based Controller examples, Minimality

- Ex 7.1 Discrete case
- Ex 7.2 Discrete case with parameters
Ex 7.1 Discrete case

Let
\[
\begin{align*}
x_1(t+1) &= 0.2x_1(t) + 0.2x_2(t) + u(t) \\
x_2(t+1) &= 0.1x_3(t) + 0.1x_2(t) \\
x_3(t+1) &= 0.4x_3(t) \\
y(t) &= x_1(t) + x_2(t)
\end{align*}
\]

be an LTI DT stochastic system

1) Analyze the structural properties, and find the unreachable and unobservable eigenvalues.

2) Is it possible, to let the state assume any arbitrary value, starting from known \( x(0) \)? If yes, how?

3) Is it possible to determine a model based controller in order to stabilize the system, with assigned closed loop eigenvalues in 0.1, 0.25, 0.01? If yes, determine it.

4) Is the system minimal?
Ex 7.1 question 1): Solution

First, notice that

\[
A = \begin{bmatrix}
0.2 & 0.2 & 0 \\
0 & 0.1 & 0.1 \\
0 & 0 & 0.1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & 1 & 0
\end{bmatrix}.
\]

Evaluating the Reachability matrix we obtain:

\[
R = \begin{bmatrix}
0.04 & 0.2 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \text{ and we notice that its rank is } 1.
\]

Being \( n = 3 \), it means that the system is not completely reachable.

The Reachable space has dimension 1, and the system will present one reachable eigenvalue and two unreachable eigenvalues.
Ex 7.1.1: Solution

- Evaluating the Observability matrix we obtain:

\[
\mathcal{O} = \begin{bmatrix}
1 & 1 & 0 \\
0.2 & 0.3 & 0.1 \\
0.04 & 0.07 & 0.07 \\
\end{bmatrix}
\]

and we notice that its rank is 3.

Being \( n = 3 \), it means that the system is completely observable.

- The system will present all three observable eigenvalues.
Ex 7.1.1: Solution

- Being the system completely observable but not reachable, it is possible, analyzing the transfer matrix, to find the unreachable eigenvalues.

- We start by evaluating the transfer function
  \[ G(z) = C(zI - A)^{-1}B + D, \]
  that will present only the reachable and observable eigenvalues in the poles:
  \[ G(z) = \frac{1}{z-0.2} \]

- Then \( \lambda = 0, 2 \) is reachable and observable, while \( \lambda = 0.1 \) and \( \lambda = 0.4 \) are unreachable and observable.

- Being all eigenvalues stable (\( |\lambda_i| < 1 \) for \( i = 1, 2, 3 \)), then system is also stabilizable and detectable.
In order to satisfy the question (reach an arbitrary value starting from $x(0)$) the system needs to be completely reachable, then the answer is no.
Ex 7.1.3: Solution

- The system presents two unreachable eigenvalues that cannot be placed anywhere, and that will be present in the final closed loop eigenvalues.
- Because they (0.1 and 0.4) are not both present in the assigned set (0.1, 0.25, 0.01), then it is not possible to design a Model based controller.
Ex 7.1.4: Solution

- The system is not minimal, because it is possible to find a system of order 1 giving raise to the same I/O representation.
- The system is not minimal, because it is not completely reachable and completely observable.
Ex 7.2 Discrete case with parameters

- Given the following LTI, TD system with real parameters $\beta$ and $\gamma$:

\[
\begin{align*}
  z_1(t + 1) &= 0.2z_1(t) + 0.2z_2(t) + u(t) \\
  z_2(t + 1) &= 0.2z_2(t) + u(t) \\
  z_3(t + 1) &= 0.3z_3(t) + \gamma u(t) \\
  y(t) &= \beta z_1(t) + z_3(t)
\end{align*}
\]

1. Study the structural properties with respect to the varying parameters.

2. If $\gamma = 1$ and $\beta = 1$, if possible, design a model based controller such that the desired eigenvalues be located in $\{0.1, 0.1, 0.1\}$ and the observer ones in $\{0, 0, 0\}$.
Ex 7.2 question 1): Solution

First, notice that

\[
A = \begin{bmatrix}
0 & 0.2 & 0 \\
0 & 0.1 & 0 \\
0 & 0 & 0.3 \\
\end{bmatrix},
B = \begin{bmatrix}
1 \\
1 \\
\gamma \\
\end{bmatrix},
C = \begin{bmatrix}
\beta & 0 & 1 \\
\end{bmatrix}.
\]

Evaluating the Reachability matrix we obtain:

\[
R = \begin{bmatrix}
0.12 & 0.4 & 1 \\
0.04 & 0.2 & 1 \\
\gamma 0.09 & \gamma 0.3 & \gamma \\
\end{bmatrix},
\]

and we notice that its determinant

\[
\det(R) = 0.020\gamma
\]

is zero only if \(\gamma = 0\). Notice that in this case, the last row is a zero row and the matrix looses rank and its rank becomes equal to 2. Being \(n = 3\), it means that the system is not completely reachable.

For \(\gamma = 0\) the Reachable space has dimension 2, and the system will present one unreachable eigenvalue and two reachable eigenvalues.

For \(\gamma \neq 0\) the system is completely reachable.
Ex 7.2.1: Solution

- Evaluating the Observability matrix we obtain:
  \[ O = \begin{bmatrix}
  \beta & 0 & 1 \\
  0.2\beta & 0.3 & 0.2 \\
  0.04 & 0.06\beta & 0.09 \\
\end{bmatrix}, \text{ and we notice that its rank is}
\]
  3 for \( \beta \neq 0 \) (completely observable) and is 2 for \( \beta = 0 \).

- For \( \beta \neq 0 \), the system will present three observable eigenvalues and for \( \beta = 0 \) it will present two observable eigenvalues because we have two column that are zeros and the rank becomes 1.
Ex 7.2.1: Solution

- Being the system not completely observable and/or not reachable depending on the parameters, it is not possible, analyzing the transfer matrix, to find immediately the unreachable and/or unobservable eigenvalues.

- But being the three eigenvalues stable, we already know that the system is stabilizable and detectable.

- In any case, we start by evaluating the transfer function \( G(z) = C(zI - A)^{-1}B + D \), that will present only the reachable and observable eigenvalues in the poles:

\[
G(z) = \frac{\beta(z-0.1)(z-0.3)+0.2\beta(z-0.3)+\gamma(z-0.2)(z-0.1)}{(z-0.2)(z-0.1)(z-0.3)} = \frac{\beta(z-0.3)(z+0.1)+\gamma(z-0.2)(z-0.1)}{(z-0.2)(z-0.1)(z-0.3)}
\]

- As expected if \( \gamma = 0 \) we have the cancellation of the \( \lambda = 0.3 \) that is the unreachable eigenvalue. While if \( \beta = 0 \) we have two eigenvalues that are unobservable \( \lambda = 0.2 \) and \( \lambda = 0.1 \).
Ex 7.1.1: Solution

- By analyzing the transfer function, is it clear that if $\beta \neq 0$ and $\gamma \neq 0$, all the eigenvalues are reachable and observable.

- If $\gamma \neq 0$, but $\beta = 0$, then $G(z) = \frac{\gamma}{z-0.3}$. Then they are all reachable, but $\lambda = 0.1$ and $\lambda = 0.2$ are not observable.

- If $\gamma = 0$, but $\beta \neq 0$, then $G(z) = \frac{\beta(z+0.1)}{(z-0.2)(z-0.1)}$, the $\lambda = 0.3$ is unreachable.

- If both $\gamma = 0$ and $\beta = 0$, then $G(z) = 0$. we only know that 1 eigenvalue is unreachable and 2 are not observable.

- In order to analyze these two cases we need to resort to modal tests. (left as homework)
Ex 7.2.2: Solution

- If $\gamma = 1$ and $\beta = 1$, the system is completely reachable, but not completely observable.
- It is then possible to design a model based controller such that the desired eigenvalues be located in \( \{0.1, 0.1, 0.1\} \) and the observer ones in \( \{0, 0, 0\} \).
- LEFT AS HOMEWORK
Thanks

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